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ON THE ABSORPTION COEFFICIENT OF A GRAY MEDIUM

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The process of absorption of radiant energy by a gray medium is considered. Formulas for the absorption coefficient of a gray medium are derived on the basis of the reciprocity principle and Planck's law.

In reference [1] the author has shown that the total molecular energy transfer vector $\mathbf{q}_m$ and the total radiant energy transfer vector $\mathbf{q}_r$ are given by

$$\mathbf{q}_m = -a_{mm}\,\mathrm{grad}\,\varepsilon_{mm} - a_{rr}\frac{\varepsilon_{mr}}{\varepsilon_{rr}}\,\mathrm{grad}\,\varepsilon_r,$$

$$\mathbf{q}_r = -a_{mm}\frac{\varepsilon_{rm}}{\varepsilon_{mm}}\,\mathrm{grad}\,\varepsilon_{mm} - a_{rr}\,\mathrm{grad}\,\varepsilon_r.$$

The molecular and radiant energy densities are

$$\varepsilon_{mm} = \rho c T, \quad \varepsilon_{лл} = 4\sigma_0 T_r^4/c_0.$$

The density of the molecular energy which is entrained by the radiant medium is

$$\varepsilon_{mr} = \rho_r c_r T_r,$$

where

$$\rho_r = 4\sigma_0 T_r^4/c_0^3.$$

If we assume that the molecular medium absorbs only the whole amount of its own radiant energy, then the density of the radiant energy which is absorbed by the molecular medium is $\varepsilon_{rm} = 4\sigma_0 T^4/c_0$.

Using the reciprocity principle, we obtain the relation

$$a_{rr}\varepsilon_{mr}/\varepsilon_{rr} = a_{mm}\varepsilon_{rm}/\varepsilon_{mm}. \tag{1}$$

This relation, with the equalities $a_{rr} = c_0/4k$, $a_{mm} = \lambda/c\rho$, yields

$$k = \rho^2 c^2 c_r T_r/16\lambda\sigma_0 T^3. \tag{2}$$

This equation relates the absorption coefficient of a gray medium k with its thermal conductivity λ.

One can introduce into the analysis the index of refraction of the medium n and assume that the molecular medium absorbs only a part of the external radiant energy of the medium.

It is known [2] that the coefficient of entrainment of radiant energy is $1 - n^{-2}$.

Under these assumptions we have

$$\varepsilon_{rr} = 4\sigma_0 T_r^4 n/c_0, \quad \varepsilon_{rm} = (1 - n^{-2})\,4\sigma_0 T_r^4 n/c_0,$$

$$\rho_r = 4\sigma_0 T_r^4 n/c_0^3, \quad a_{rr} = c_0/4kn.$$

Substituting these expressions into (1) we obtain

$$k = \rho^2 c^2 c_r T/16\,\lambda\sigma_0 T_r^3 (n^2 - 1). \tag{3}$$

At thermodynamic equilibrium ($T = T_r$) equation (3) reduces to

$$k = \rho^2 c^3/16\lambda\sigma_0 T^2 (n^2 - 1). \tag{4}$$

In our opinion equation (3) gives a better representation of the physical nature of the "entrainment" of radiant energy by a gray gas than equation (2). However, equation (3) involves the index of refraction of the medium, which is not accurately known.

Equations (2) and (3) can be used only in those cases in which heat is transferred in the medium solely by radiation and conduction. Those are the cases of flow of a gray medium near solid surfaces.

When heat is transferred in the medium by radiation, convection and conduction, the principle of reciprocity leads to very cumbersome formulas for the determination of k, which cannot be used in practice.

Thus it is necessary to find a more effective method for the determination of k than the method based on the principle of reciprocity.

Consider a radiation field in thermodynamic equilibrium with a stationary gray medium.

The density of monochromatic equilibrium radiation $d\varepsilon_r$ is

$$d\varepsilon_r = 4\pi I_\nu n c_0^{-1} d\nu.$$

The distribution function I_ν is given by Planck's law [3]:

$$I_\nu = \frac{2h\nu^3 n^2}{c_0^2}\left(e^{\frac{h\nu}{k_0 T}} - 1\right)^{-1}.$$

Consequently,

$$d\varepsilon_r = \frac{8\pi h\nu^3 n^3}{c_0^3}\,\frac{1}{e^{\frac{h\nu}{k_0 T}} - 1}\,d\nu.$$

The radiation field in the gray medium is perfectly black. It contains photons with frequencies from 0 to ∞. Therefore the global radiation density is

$$\varepsilon_r = \frac{8\pi h n^3}{c_0^3}\int_0^\infty \frac{\nu}{e^{\frac{h\nu}{k_0 T}} - 1}\,d\nu,$$

or

$$\varepsilon_r = \frac{8\pi^5 k_0^4 n^3}{15 c_0^3 h^3}T^4. \tag{5}$$

We may also write

$$\varepsilon_r = 4\pi I_0 n/c_0, \tag{6}$$

where

$$I_0 = \sigma_0 T^4/\pi,$$

and

$$\sigma_0 = 2\pi^5 k_0^4 n^2/15 c_0^2 h^3. \tag{7}$$

The concentration of photons with frequency ν is $dn_0 = d\varepsilon_r/h\nu$, or

$$dn_0 = \frac{8\pi\nu n^3}{c_0^3}\,\frac{1}{e^{\frac{h\nu}{k_0 T}} - 1}\,d\nu.$$

The concentration of photons in the frequency range 0 to ∞ is

$$n_0 = \frac{8\pi n^3}{c_0^3}\int_0^\infty \frac{\nu^2\,d\nu}{e^{\frac{h\nu}{k_0 T}} - 1}.$$

Introducing the variable $x = h\nu/k_0 T$, we can write

$$n_0 = \frac{8\pi k_0^3 T^3 n^3}{c_0^3 h^3}\int_0^\infty \frac{x^2 dx}{e^x - 1}.$$

Taking into account that

$$\frac{1}{e^x - 1} = e^{-x} + e^{-2x} + e^{-3x} + \dots,$$

we have

$$\int_0^\infty \frac{x^2\,dx}{e^x - 1} = \int_0^\infty e^{-x} x^2\,dx + \int_0^\infty e^{-2x} x^2 dx + \int_0^\infty e^{-3x} x^2 dx + \dots =$$

$$= 2\left(1 + \frac{1}{2^3} + \frac{1}{3^3} + \dots\right) = 2.332.$$

Substituting this result, we obtain

$$n_0 = 18.59\pi\,(k_0 n/c_0 h)^3 T^3. \tag{8}$$

Introducing the notion of a mean photon energy ε_0, defined by the equation $\varepsilon_0 = \varepsilon_r/n_0$, or

$$\varepsilon_0 = \frac{8\pi^5 k_0^4 n^3}{15 c_0^3 h^3} T^4 \Big/ 18.59\pi \left(\frac{k_0 n}{c_0 h}\right)^3 T^3.$$

Carrying out the arithmetic, we obtain

$$\varepsilon_0 = 2.8 k_0 T. \tag{9}$$

Equation (9) shows that the mean photon energy depends only on the equilibrium temperature of the medium.

Assume that emission and absorption of radiant energy by the equilibrium medium take place in a discrete manner, in the form of photons, each of which has the energy ε_0.

The volume of a space cell occupied by a photon is

$$v_0 = \varepsilon_0/\varepsilon_r = 1/n_0.$$

Taking account of equation (8), we obtain

$$v_0 = \frac{1}{18.59\pi} \frac{c_0^3 h^3}{k_0^3 T^3 n^3} = 0.0171 \frac{c_0^3 h^3}{k_0^3 T^3 n^3}. \tag{10}$$

Let us choose a direction **s** in the medium at equilibrium, and let l_{rm} be a line segment in this direction. Let this segment be the minimum length over which the total energy emitted by the medium in the **s** direction per unit solid angle is the same as that emitted by a perfectly black body. This definition allows us to write

$$\eta\, l_{rm} = I_0. \tag{11}$$

In thermodynamic equilibrium, radiant energy with intensity I_0 must be fully absorbed over the distance l_{rm}, as otherwise the intensity at the end point of the segment l_{rm} would differ from I_0. Using Bouguer's law, we can write

$$k I_0 l_{rm} = I_0,$$

where $k = 1/l_{rm}$ is the coefficient of absorption of the medium.

The emission of the space cell v_0 of a photon is perfectly black. Thus

$$4\pi\eta\, v_0 = 4\pi\, l_{rm}^2\, I_0.$$

Taking into account that $I_0/\eta = l_{rm}$, we obtain $v_0 = l_{rm}^3$, which gives

$$l_{rm} = 0.258\, c_0 h/n k_0 T. \tag{12}$$

Thus, the volume of the space cell of a photon is a cube with edges of length l_{rm}.

The coefficient of absorption k is thus given by the equation

$$k = 3.88\, n k_0 T/c_0 h. \tag{13}$$

Equation (13) shows that the absorption coefficient for a medium at equilibrium is a function of its optical properties (index of refraction n) and its temperature T.

Note, that for a vacuum $n = 1$ and $k_0 = 0$. Consequently, $k = 0$.

The time of formation of a photon τ_0 is determined by the equation

$$4\sigma_0 T^4 v_0 k \tau_0 = 2.8 k_0 T,$$

which gives

$$\tau_0 = 2.8 k_0 T / 4\sigma_0 T^4 v_0 k.$$

Taking account of equations (10) and (13), we obtain

$$\tau_0 = 0.258 h / k_0 T. \tag{14}$$

Equation (14) indicates that the time of formation of a photon is independent of the optical properties of the medium (the index of refraction n) and depends only on its temperature T.

Equations (12) and (14) yield $l_{rm} = c_0 \tau_0 / n$, i.e., the photon mean free path is equal to the distance travelled by the radiant energy during the time of formation of one photon.

As the time of formation of a photon τ_0 does not depend on n, we have the relation

$$l_{rm} n / c_0 = \text{const.} \tag{15}$$

In a medium at equilibrium we have the relation $\eta = k\sigma_0 T^4/\pi$, or

$$\eta = 50.7 n^3 k_0^5 T^5 / c_0^3 h^4. \tag{16}$$

Equation (16) can be used to calculate the true emission coefficient of a medium at equilibrium.

Consider a gray medium in a state of nonequilibrium. The radiation field of such a medium is characterized by the true emission intensity I_0 and the external radiation intensity I_r. There exists a length l_{rr} over which the radiant energy with intensity I_r is fully absorbed by the medium, which allows us to write $kI_r l_{rr} = I_r$ and hence

$$k = 1 / l_{rr}. \tag{17}$$

Over the length l_{rm} the medium absorbs the radiant energy $kI_r l_{rm}$. Under conditions of local radiative equilibrium this energy is transformed by the medium into true emission energy with intensity I_0. Thus $kI_r l_{rm} = I_0$, but taking into account that $I_0/I_r = (T/T_r)^4$, we obtain

$$k = 3.88 \frac{n k_0 T}{c_0 h} \left(\frac{T}{T_r} \right)^4. \tag{18}$$

Equation (18) shows that the quantity k depends on the ratio T/T_r, which characterizes the degree of equilibrium of a gray medium, and on the direction for which the quantity T_r is evaluated.

Averaging the variable k over all directions, we obtain

$$k = 3.88 \frac{n k_0 T^5}{c_0 h} \frac{1}{4\pi} \int_{4\pi} T_r^{-4} \, d\omega,$$

which is approximately equal to

$$k = 3.88 \frac{n k_0 T}{c_0 h} \left(\frac{T}{T_r} \right)^4. \tag{19}$$

The time of formation of a photon in a nonequilibrium medium is determined by the equation

$$4\sigma_0 T_r^4 v_0 k \tau_r = 2.8 k_0 T,$$

which gives

$$\tau_r = 2.8 k_0 T / 4\sigma_0 T_0^4 v_0 k.$$

Taking account of (18), we can write

$$\tau_r = \frac{2{,}8 k_0 T c_0 h}{4\sigma_0 T^4 v_0 \, 3{,}88 n k_0 T},$$

or

$$\tau_r = \tau_0,$$

i.e., the times of formation of a photon in equilibrium and nonequilibrium media are the same.

Equation (19) can be used in the analysis of complex heat transfer during the motion of a gray continuum near a heat-absorbing wall with temperature T_W. Assuming that the medium is in local radiative equilibrium, we can rewrite equation (19) in the form

$$k = 3.88 \frac{n k_0 T}{c_0 h} \frac{T^4}{T^4 + \Delta T^4}.$$

Taking into account that

$$\Delta T^4 = \frac{1}{k} \frac{dT^4}{dy},$$

we obtain

$$k = 3.88 \frac{n k_0 T}{c_0 h} - \frac{4}{T} \frac{dT}{dy}. \qquad (20)$$

In the case of linear dependence of $T - T_W$ on y, equation (20) for an equilibrium layer with temperature T_δ becomes

$$k = 3.88 \frac{n k_0 T_\delta}{c_0 h} - \frac{4}{T_\delta} \frac{T_\delta - T_W}{l_s} \qquad (21)$$

In the case of a layer at equilibrium we have

$$q_r = -\frac{\sigma_0}{k} \left| \frac{dT^4}{dy} \right|_\delta = \sigma_0 \left(\frac{1}{A_W} - \frac{1}{2} \right)^{-1} (T_\delta^4 - T_W^4).$$

Hence

$$k l_s = 4 \left(\frac{1}{A_W} - \frac{1}{2} \right) \left\{ \left(\frac{T_W}{T_\delta} + 1 \right) \left[\left(\frac{T_W}{T_\delta} \right)^2 + 1 \right] \right\}^{-1}. \qquad (22)$$

The specific total heat flux q consists of a radiant heat flux and a conductive heat flux, and in optically dense media is defined by

$$q = \sigma_0 \left(\frac{1}{A_W} - \frac{1}{2} \right)^{-1} (T_\delta^4 - T_W^4) + \frac{\lambda}{l_s} (T_\delta - T_W). \qquad (23)$$

Equations (21)-(23) involve four unknown variables – q, T_δ, k, and l_s. Given one of these, the other three can be determined.

Thus, these equations can be used in the analysis of various heat exchangers with complex heat transfer processes.

NOTATION

n – index of refraction of the medium, depending on its optical properties; c_0 – speed of light; ν – vibration frequency; η – true emission coefficient of the medium; l_{rm} – photon mean free path; T_δ – temperature of an equilibrium layer; l_s – distance between the equilibrium layer and the wall.

REFERENCES

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